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A single action for the scalar–tensor theory of gravity

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Abstract. The standard form of the scalar–tensor theory gives eleven equations for eleven unknowns, the metric tensor g_{ij} and the scalar field ψ. Here we eliminate the scalar field to produce a theory that has just ten equations for ten unknown g_{ij}. The resulting expression for the action of fields and matter is contained completely in a single expression.

1. Introduction

The scalar–tensor theory developed by Jordan (1955) and Brans and Dicke (1961) can be derived from a variational principle with an action (Dicke 1964)

$$\mathscr{A} = \int (\psi^2 R + \omega\psi^{,i}\psi_{,i} + \nu\mathscr{L})\sqrt{(-g)}\, d^4x \tag{1}$$

where ω is an arbitrary coupling constant and ν a constant that normalises the masses. For matter only

$$\mathscr{L}\sqrt{(-g)} = \sum_i \int \delta^4(x-z)m_i\, ds \tag{2}$$

where

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{3}$$

with g_{ij} the metric tensor and m_i the particle masses.

In this representation, g_{ij} is the geodesic metric, that is, particles and light rays follow geodesics of the geometry; the field equations are not those of general relativity since they contain the scalar field ψ which is determined by the geometry of the g_{ij} and the energy–momentum tensor T_{ij}. The theory has eleven equations, given by varying the action (1) with respect to the ten g_{ij}, and ψ, to determine the eleven variables g_{ij} and ψ in terms of the distribution of the sources.

In principle, we might expect to be able to eliminate ϕ and reduce the theory to ten equations for ten g_{ij}, and this is the problem considered in this paper. It is indeed possible to do this, not in terms of the geodesic space g_{ij}, but in terms of a conformally related space $g^*_{ij} = g_{ij}/\phi$.

2. Conformal representation

While the above representation of the scalar–tensor theory is in terms of the geodesic metric g_{ij}, we can equally write the theory in terms of a conformally related metric. This can be considered either as a re-scaling of units of time (and therefore length) or simply as an alternative mathematical representation of the theory. In this new metric, particles no longer follow geodesics of the geometry g^*_{ij}, but their (non-geodesic) trajectories are still fully determined by the transformed equations. We do not transform the particle masses m_i, the mass is considered here to be a pure number characterising the particle (e.g. a number times some standard mass such as the proton), a particle containing m_i protons and electrons has the same number independent of the units for measuring space and time. We therefore consider the action (1) in the new metric g^*_{ij} where

$$g_{ij} = \phi g^*_{ij} \tag{4}$$

and under such a transformation we have (cf Synge 1960)

$$\sqrt{(-g)} = \phi^2 \sqrt{(-g^*)}, \qquad \mathscr{L}\sqrt{(-g)} = \phi^{1/2}\mathscr{L}^*\sqrt{(-g^*)} \tag{5}$$

$$R = \frac{1}{\phi}R^* + 3g^{*ij}\frac{\phi_{;ij}}{\phi^2} - \frac{3}{2}g^{*ij}\frac{\phi_{,i}\phi_{,j}}{\phi^3}. \tag{6}$$

The action (1) becomes

$$\mathscr{A} = \int \left(\psi^2\phi R + 3\psi^2\Box\phi - \frac{3}{2}\psi^2\frac{\phi^{,i}\phi_{,i}}{\phi} + \omega\phi\psi^{,i}\psi_{,i} + \nu\phi^{1/2}\mathscr{L}\right)\sqrt{(-g)}\,\mathrm{d}^4x \tag{7}$$

where for convenience we have dropped the asterisk. Let us now take $\phi = \psi^{2/n}$ so that

$$\mathscr{A} = \int \left(\phi^{n+1}R + \frac{3\Box\phi^{n+1}}{n+1} + \frac{1}{4}(n^2\omega - 12n - 6)\phi^{n-1}\phi^{,i}\phi_{,i} + \nu\phi^{1/2}\mathscr{L}\right)\sqrt{(-g)}\,\mathrm{d}^4x. \tag{8}$$

$\Box\phi\sqrt{(-g)}$ is a perfect differential and so makes no contribution to the field equations obtained by varying the action; if we now choose n such that

$$n^2\omega - 12n - 6 = 0 \qquad n = \frac{6 \pm 6(1+\omega/6)^{1/2}}{\omega} \tag{9}$$

the action is just

$$\mathscr{A} = \int (\phi^{n+1}R + \nu\phi^{1/2}\mathscr{L})\sqrt{(-g)}\,\mathrm{d}^4x. \tag{10}$$

3. Elimination of the scalar field

Varying the above action with respect to ϕ gives

$$\phi^{(2n+1)/2} = -\frac{\nu}{2(n+1)}\frac{\mathscr{L}}{R} \tag{11}$$

and variation with respect to g_{ij} gives

$$\phi^{n+1}(R^{ij} - \tfrac{1}{2}g^{ij}R) - g^{ij}\Box\phi^{n+1} + g^{ik}g^{jm}(\phi^{n+1})_{;km} = \nu\phi^{1/2}T^{ij}. \tag{12}$$

Consequently the scalar field can be completely eliminated from the field equations in favour of $(\mathscr{L}/R)$. For a pure matter action $\mathscr{L}$ given by (2) is homogeneous of degree 1/2 in g_{ij}, hence

$$g_{ij}\frac{\partial\mathscr{L}}{\partial g_{ij}} = \tfrac{1}{2}\mathscr{L} = g_{ij}\tfrac{1}{2}T^{ij} = \tfrac{1}{2}T \tag{13}$$

so $\mathscr{L} = T$. The field equations are then

$$R_{ij} - \frac{1}{2} g_{ij} R = -2\nu(n+1)\frac{R}{T} T_{ij} + \left(\frac{R}{T}\right)^{(2n+2)/(2n+1)} (g^{ik}g^{jm} - g^{ij}g^{km})\left[\left(\frac{T}{R}\right)^{(2n+2)/(2n+1)}\right]_{;km}. \tag{14}$$

4. The single action

Since the variation of the action (10) gives ϕ we may replace ϕ inside the action by its value (11) so that, apart from numerical constants,

$$\mathscr{A} = \int \left(\frac{\mathscr{L}}{R}\right)^{(2n+2)/(2n+1)} R\surd(-g)\, d^4x \tag{15}$$

where

$$n = 6[1 \pm (1+\omega/6)^{1/2}]/\omega. \tag{16}$$

This procedure of replacing ϕ inside the action is legitimate since the action (10) contains no derivatives of ϕ and the field equations obtained by varying (15) are just (14), which in turn is just the scalar–tensor theory in a different (conformally related) geometry. We have therefore succeeded in eliminating the scalar field altogether and the action (15) is all that is needed to specify the theory, which now has ten equations for ten unknowns. Moreover the fact that the theory can be condensed into a single action is not discouraging for attempts to relate the scalar–tensor theory to Mach's principle.

In the limit $\omega \to \infty$ the one-body tests of the scalar–tensor theory are the same as in general relativity. However, as is clear from the action (15), the theory is not the same as general relativity. From equation (9) we have $n \to 0$ as $\omega \to \infty$ and the action (15) reduces to

$$\mathscr{A} = \int \frac{\mathscr{L}^2}{R} \surd(-g)\, d^4x. \tag{17}$$

This limit is interesting in its own right as an extension of general relativity, so further discussion is delayed for a subsequent publication (Roxburgh 1977).

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